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# Flag curvature of invariant Randers metrics on homogeneous manifolds

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**Abstract**

In this paper we review the recent results about the flag curvature of invariant Randers metrics on homogeneous manifolds. By using some counterexamples we show that the formula obtained for the flag curvature of these metrics is incorrect. Then we give an explicit formula for the flag curvature of invariant Randers metrics on naturally reductive homogeneous manifolds $(G/H, g)$, where the Randers metric is induced by the invariant Riemannian metric $g$ and an invariant vector field $\tilde{X}$ which is parallel with $g$.

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## 1. Introduction

The geometry of Finsler manifolds is one of the interesting subjects in differential geometry which has many physical applications (for example, see [1] and [2]). One of the important quantities which associate with a Finsler metric is the flag curvature which is a generalization of the concept of sectional curvature in Riemannian geometry.

But, in general, the computation of the flag curvature of a Finsler metric is very difficult. Therefore, it is very important to find an explicit and applicable formula for the flag curvature. In this paper we want to find such an explicit formula for the flag curvature of a special type of invariant Randers metrics on homogeneous manifolds.

Deng and Hou studied invariant Finsler metrics on reductive homogeneous manifolds and gave an algebraic description of these metrics and obtained a necessary and sufficient condition for a homogeneous manifold to have invariant Finsler metrics (see [4]). Also they studied invariant Randers metrics on homogeneous Riemannian manifolds and used this structure to construct Berwald space which is neither Riemannian nor locally Minkowskian (for more details see [5]). They gave a formula for the flag curvature of invariant Randers metrics on homogeneous manifolds in [5].

  

In this paper by using some counterexamples we show that the formula which was obtained in [5] (see [5], theorem 3.2.) is incorrect. Also we explain why this formula [5] and an example which is given in [4] (see [4], p 8251) are incorrect. Then we give an explicit formula for the flag curvature of invariant Randers metrics on naturally reductive homogeneous manifolds $(G/H, g)$, where the Randers metric is induced by the invariant Riemannian metric $g$ and an invariant vector field $\tilde{X}$ which is parallel with $g$.

## 2. Preliminaries

**Definition 2.1** (see [6] or [8]). *A homogeneous space $G/H$ of a connected Lie group $G$ is called reductive if the following conditions are satisfied.*

*(1) In the Lie algebra $\mathfrak{g}$ of $G$ there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (direct sum of vector subspaces).*

*(2) ad $(h)\mathfrak{m} \subset \mathfrak{m}$, for all $h \in H$,*
*where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the identity component $H_0$ of $H$. Also ad $(h)$ denotes the adjoint representation of $H$ in $\mathfrak{g}$.*
*Note that condition (2) implies*

*(2)′ $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$*
*and conversely if $H$ is connected, then (2)′ implies (2). For example, $G/H$ is reductive in either of the following cases (see [8]):*

- *$H$ is compact,*
- *$H$ is connected and semi-simple,*
- *$H$ is a discrete subgroup of $G$; $\mathfrak{h} = 0$ and $\mathfrak{m} = \mathfrak{g}$.*

Let $G/H$ be a reductive homogeneous manifold with the invariant Riemannian metric $g$ where the subspace $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ with respect to the inner product on $\mathfrak{g}$. Also let

$$V = \{X \in \mathfrak{m} \mid \mathrm{ad}(h)X = X, \langle X, X\rangle < 1, \forall h \in H\},$$

where $\langle\, ,\rangle$ is the inner product induced by $g$. Then for any $X \in V$ there exists an invariant Randers metric on $G/H$ by the following formula (see [5]):

$$F_X(xH, y) = \sqrt{g(xH)(y, y)} + g(xH)(\tilde{X}, y), \qquad y \in T_{xH}(G/H), \tag{2.1}$$

where $\tilde{X}$ is the corresponding invariant vector field on $G/H$ to $X$.

In the following we state the theorem given in [5] (see [5], theorem 3.1).

**Theorem 2.2.** *Let $Y$ be a nonzero vector in $\mathfrak{m}$ and $P$ be a plane in $\mathfrak{m}$ containing $Y$. Then the flag curvature of the flag $(P, Y)$ in $T_0(G/H)$ is given by*

$$K(P, Y) = \frac{2\sqrt{g(Y, Y)}}{2\sqrt{g(Y, Y)} + g(X, Y)} K(P),$$

*where $K(P)$ is the Riemannian curvature of $P$ of the Riemannian metric $g$, and*

$$K(P) = \frac{g([[Y, U]_{\mathfrak{h}}, Y], U)}{g(U, U)g(Y, Y) - g^2(U, Y)},$$

*where $U$ is any vector in $P$ such that* span$\{Y, U\} = P$ *and $[Y, U]_{\mathfrak{h}}$ is the orthogonal projection of $[Y, U]$ to $\mathfrak{h}$.*

We will show that theorem 2.2 is wrong.

## 3. Counterexamples

Let $G$ be a connected Lie group and $H = \{e\}$, then $G/H = G$. Thus, $G/H$ is a reductive homogeneous manifold with $\mathfrak{m} = \mathfrak{g}$ and $\mathfrak{h} = 0$. Also let $g$ be an invariant Riemannian metric on $G/H = G$ and $0 = X \in \mathfrak{m} = \mathfrak{g}$.

In this case the invariant Randers metric defined by $g$ and $X$ is Riemannian, because

$$F_X(xH, y) = \sqrt{g(xH)(y, y)} + g(xH)(0, y) = \sqrt{g(x)(y, y)}.$$

Also we know that if our Finsler metric be Riemannian then the flag curvature reduces to the familiar sectional curvature [3].

Under the above conditions, since $\mathfrak{h} = 0$,

$$[Y, U]_{\mathfrak{h}} = 0.$$

Hence,

$$g([[Y, U]_{\mathfrak{h}}, Y], U) = g([0, Y], U) = g(0, U) = 0.$$

Therefore, $K(P) = 0$ and we obtain $K(P, Y) = 0$.

Since, in this case the flag curvature is the same sectional curvature of $g$, we obtain the following incorrect proposition.

**Proposition 3.1.** *The sectional curvature of all Lie groups with invariant Riemannian metrics is zero.*

Obviously proposition 3.1 is wrong. In the following we give some counterexamples for this proposition that show there exist many Lie groups with invariant Riemannian metrics and nonzero sectional curvatures.

The simplest counterexample is the standard metric on the Lie group $S^3 = SU(2)$ which has the curvature 1.

Another counterexamples can be obtained by the following theorem.

**Theorem 3.2** (see [10]). *Let $G$ be a connected nilpotent Lie group. Let $B$ be a positive definite symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of $G$, and let $M$ be the Riemannian manifold obtained by left translation of $B$ to every tangent space of $G$. Then these are equivalent.*

- *$M$ has a positive sectional curvature.*
- *$M$ has a negative sectional curvature.*
- *$G$ is not commutative.*

So if we consider a connected nilpotent noncommutative Lie group $G$ with an invariant Riemannian metric, then it has a nonzero sectional curvature. (One can find another counterexamples in [7].)

By attention to the above counterexamples we have shown that proposition 3.1 and therefore theorem 2.2 are incorrect.

## 4. Flag curvature of invariant Randers metrics on homogeneous manifolds

But why is theorem 2.2 incorrect?

If we see the proof of this theorem 2.2 we see that the authors in [5] used the following formula

$$R(U, V)W = -[[U, V]_{\mathfrak{h}}, W] \tag{4.2}$$

for the curvature tensor of $F$ (and $g$). But equation (4.2) is a formula for tensor curvature of a special type of invariant affine connections called canonical affine connection of the second kind (see theorem 10.3 in [8] or theorem 2.6 in [6], p 193) which is not the Riemannian connection. Thus, we cannot use equation (4.2) for curvature tensor of the Riemannian connection.

Another problem of the proof of theorem 2.2 is the formula of $g_Y(U, V)$. The formula

$$g_Y(U, V) = g(U, V)\left(1 + \frac{g(X, Y)}{2\sqrt{g(Y, Y)}}\right)$$

which is used in the proof of theorem 2.2 is not correct.

Another significant mistake in [5] is that the authors of this paper thought that all the Randers metrics discussed in [5] are of Berwald type which is wrong. In fact there is a mistake in theorem 2.2 of [5]. In general, the Randers metrics obtained in this theorem are not of Berwald type. If we want this Randers metric to be of Berwald type, we must add the condition 'suppose that the vector field $\tilde{X}$ is parallel with respect to $\tilde{a}$ where $\tilde{X}$ is the corresponding invariant vector field to $X$' to theorem 2.2 of [5]. But without this hypothesis we cannot say that the corresponding Randers metric to $\tilde{X}$ is of Berwald type.

The authors of paper [5] used equation (4.2) to compute the flag curvature of an example in paper [4] (the example after theorem 2.1 in [4], p 8251). Since equation (4.2) is not the curvature tensor of the Riemannian connection, the flag curvature of this example is not correct.

Now by the above discussion, we compute the flag curvature of invariant Randers metrics on homogeneous manifolds.

**Definition 4.1** (see [6])**.** *A homogeneous manifold $M = G/H$ with a $G$-invariant indefinite Riemannian metric $g$ is said to be naturally reductive if it admits an ad$(h)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ satisfying the condition*

$$B(X, [Z, Y]_{\mathfrak{m}}) + B([Z, X]_{\mathfrak{m}}, Y) = 0 \qquad \text{for} \quad X, Y, Z \in \mathfrak{m},$$

*where $B$ is the bilinear form on $\mathfrak{m}$ induced by $\mathfrak{g}$ and $[\,,\,]_{\mathfrak{m}}$ is the projection to $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.*

**Theorem 4.2.** *Let $G/H$ be a homogeneous manifold with the invariant Riemannian metric $g$, and $F$ be an invariant Randers metric defined by the ad $(h)$-invariant vector $X$ as follows:*

$$F(xH, Y) = \sqrt{g(xH)(Y, Y)} + g(xH)(\tilde{X}, Y),$$

*where $g(X, X) < 1$, $Y \in T_{xH}G/H$, and $\tilde{X}$ is the corresponding invariant vector field on $G/H$ to $X$. Also suppose that the vector field $\tilde{X}$ is parallel with respect to $g$ and $(G/H, g)$ is naturally reductive. Then the flag curvature of the flag $(P, Y)$ in $T_H(G/H)$ is given by*

$$K(P, Y) = \frac{A}{B - C},$$

*where*

$$A = g(\alpha, U) + g(X, \alpha)\cdot g(X, U) - \frac{g(X, Y)\cdot g(Y, U)\cdot g(Y, \alpha)}{g(Y, Y)^{\frac{3}{2}}}$$
$$+ \frac{1}{\sqrt{g(Y, Y)}}\{g(X, \alpha)\cdot g(Y, U) + g(X, Y)\cdot g(\alpha, U) + g(X, U)\cdot g(Y, \alpha)\},$$

$$B = \{g(Y,Y) + g^2(X,Y) + 2g(X,Y)\sqrt{g(Y,Y)}\}\left\{g(U,U) + g^2(X,U) - \frac{1}{\sqrt{g(Y,Y)}} \times \left\{\frac{g(X,Y)\cdot g^2(Y,U)}{g(Y,Y)} + g(X,Y)\cdot g(U,U) + 2g(X,U)\cdot g(Y,U)\right\}\right\}$$

*and*

$$C = \left\{g(Y,U)\left(1 + \frac{g(X,Y)}{\sqrt{g(Y,Y)}}\right) + g(X,U)(g(X,Y) + \sqrt{g(Y,Y)})\right\}^2,$$

*where $U$ is any vector in $P$ such that* $\mathrm{span}\{Y,U\} = P$ *and $\alpha$ in $A$ is defined by*

$$\alpha = \tfrac{1}{4}[Y,[U,Y]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[U,Y]_{\mathfrak{h}}].$$

*Note that $[\,,\,]_{\mathfrak{m}}$ and $[\,,\,]_{\mathfrak{h}}$ are the projections of $[\,,\,]$ to $\mathfrak{m}$ and $\mathfrak{h}$, respectively.*

**Proof.** Since, $\tilde{X}$ is parallel with respect to $g$, it follows from lemma 2.1 in [5] that $F$ is of Berwald type. Also the Chern connection of $F$ and the Riemannian connection of $g$ coincide (see [3], p 305); therefore, we have

$$R^F(U,V)W = R^g(U,V)W,$$

where $R^F$ and $R^g$ are the curvature tensors of $F$ and $g$, respectively. Now let $R := R^F = R^g$. Notice $g$ is naturally reductive; thus by using proposition 3.4 in [6] (p 202) we have

$$(R(U,V)W)_0 = \tfrac{1}{4}[U,[V,W]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{1}{4}[V,[U,W]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{1}{2}[[U,V]_{\mathfrak{m}},W]_{\mathfrak{m}} - [[U,V]_{\mathfrak{h}},W] \qquad \text{for} \quad U,V,W \in \mathfrak{m}.$$

Then

$$(R(U,Y)Y)_0 = \tfrac{1}{4}[U,[Y,Y]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{1}{4}[Y,[U,Y]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{1}{2}[[U,Y]_{\mathfrak{m}},Y]_{\mathfrak{m}} - [[U,Y]_{\mathfrak{h}},Y] = \tfrac{1}{4}[Y,[U,Y]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[U,Y]_{\mathfrak{h}}].$$

On the other hand,

$$K(P,Y) = \frac{g_Y(R(U,Y)Y,U)}{g_Y(Y,Y)\cdot g_Y(U,U) - g_Y^2(Y,U)}, \tag{4.3}$$

where

$$g_Y(U,V) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\{F^2(Y+sU+tV)\}|_{s=t=0} = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\{g(Y+sU+tV, Y+sU+tV) + g^2(X, Y+sU+tV) + 2\sqrt{g(Y+sU+tV,Y+sU+tV)}\,g(X,Y+sU+tV)\}|_{s=t=0}.$$

By a direct computation we get

$$g_Y(U,V) = g(U,V) + g(X,U)\cdot g(X,V) - \frac{g(X,Y)\cdot g(Y,V)\cdot g(Y,U)}{g(Y,Y)^{\frac{3}{2}}} + \frac{1}{\sqrt{g(Y,Y)}}\{g(X,U)\cdot g(Y,V) + g(X,Y)\cdot g(U,V) + g(X,V)\cdot g(Y,U)\}.$$

Therefore,

$$g_Y(Y,Y) = g(Y,Y) + g(X,Y)(g(X,Y) + 2\sqrt{g(Y,Y)}),$$

$$g_Y(U,U) = g(U,U) + g^2(X,U) - \frac{g(X,Y)\cdot g^2(Y,U)}{g(Y,Y)^{\frac{3}{2}}} + \frac{1}{\sqrt{g(Y,Y)}}\{g(X,Y)\cdot g(U,U) + 2g(X,U)\cdot g(Y,U)\},$$

$$g_Y(Y,U) = g(Y,U)\left(1 + \frac{g(X,Y)}{\sqrt{g(Y,Y)}}\right) + g(X,U)(g(X,Y) + \sqrt{g(Y,Y)}),$$

and

$$g_Y(R(U,Y)Y,U) = g_Y(\alpha,U),$$

where $\alpha = \frac{1}{4}[Y,[U,Y]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[U,Y]_{\mathfrak{h}}]$.

Substituting the above formulae in equation (4.3) completes the proof. □

**Remark 4.3.** In the formula obtained in theorem 4.2 if we assume that $U$ is orthogonal to $Y$ with respect to $g$, then we can obtain a simpler formula.

**Corollary 4.4.** *Assume a Lie group $G$ as a reductive homogeneous space with $H = \{e\}$, $\mathfrak{h} = \{0\}$, and $\mathfrak{m} = \mathfrak{g}$. Then our formula for the flag curvature is simpler because in this case we have*

$$\alpha = \frac{1}{4}[Y,[U,Y]].$$

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